9.3 - 9.5 Techniques of Convergence of series

* N^{th} test of divergence:

Let $\sum a_n$ be any series. If $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges

(ie $\sum a_n \not\to$)

Eg: $\sum\limits_{n=1}^{\infty} \frac{n}{n+1}$; Show that the series diverge!

ANS: Let $a_n = \frac{n}{n+1}$

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right) = \lim_{n\to\infty}\left(\frac{n}{n}\right) = 1 \neq 0$$ so, $\sum \frac{n}{n+1}$ diverges

Note: Even though $\lim\limits_{n\to\infty}\left(\frac{1}{n}\right) = 0$, $\sum \frac{1}{n}$ diverges!

I - Convergence test of positive series

① Integral test

Suppose $a_n > 0$ and consider the series $\sum a_n$ - Let $f(n) = a_n$, where f is a positive, continuous and decreasing function. If $\int_1^{\infty} f(x)\,dx \longrightarrow$, $\sum a_n \longrightarrow$

If $\int_1^{\infty} f(x)\,dx \not\longrightarrow$, $\sum a_n \not\longrightarrow$

Eg: Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

(Harmonic series)

Let $a_n = \frac{1}{n}$, define $f(x) = \frac{1}{x}$, with

$f(n) = a_n = \frac{1}{n}$

So $\sum_{n=1}^{\infty} a_n \longleftrightarrow \int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x}\,dx$

$= \lim_{b\to\infty} \Big[\ln|x| \Big]_1^b = \lim_{b\to\infty} \ln|b| - 0 = \infty$

Conclusion: $\int_1^{\infty} \frac{1}{x}\,dx$ diverges, so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges!

* Always test $f(x)$: before using the intergral test.

Note: . $f(x) = \frac{1}{x}$ is continuous over $(0, \infty)$

. $f'(x) = -\frac{1}{x^2} < 0 \rightarrow f$ is decreasing over $(0, \infty)$

. $f(x) = \frac{1}{x} > 0$, for all $x \geq 1$

Ex: Discuss the convergence or divergence of p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p \neq 1$

(When $p = 1 \longrightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series)

ANS: Let $f(x) = \frac{1}{x^p} \longrightarrow \int_1^{\infty} \frac{1}{x^p}\,dx$

We shown in the section of improper integral that

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & p > 1 \\ \infty & p \leq 1 \end{cases}$$

So $$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \frac{1}{p-1} & \text{converges if } p > 1 \\ \infty & \text{diverges if } p \leq 1 \end{cases}$$

Eg: $p = 2$, (p-series test)

$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ $\underline{\underline{\text{converges}}}$!

$p = \frac{1}{3}$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \cdots$ diverges!

② Limit comparison test

Let $a_n > 0$, $b_n > 0$ and consider

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$$

- If $L = 0$ and $\sum b_n \to$, then $\sum a_n \to$
- If $L = \infty$ and $\sum a_n \to$, then $\sum b_n \to$
- If $L > 0$ then $\sum a_n \to$ if and only if $\sum b_n \to$

Eg: $$\sum_{n=1}^{\infty} \frac{1}{n^4 - n + 1}$$

Let $a_n = \dfrac{1}{n^4 - n + 1}$, $b_n = \dfrac{1}{n^4}$

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{n^4-n+1}}{\frac{1}{n^4}}\right) = \lim_{n\to\infty}\left(\frac{n^4}{n^4-n+1}\right) =$$

$$\lim_{n\to\infty}\left(\frac{n^4}{n^4}\right) = 1 = L > 0$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^4} \longrightarrow$ by p-series test with $p = 4 > 1$

we can conclude that $\sum_{n=1}^{\infty} \frac{1}{n^4 - n - 1}$ converges by a <u>comparison limit test</u>

Reminder: L'Hopital Rule!

Suppose $\lim_{n\to\infty} \frac{f(x)}{g(x)} \longrightarrow \frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty-\infty$
(indeterminate cases!)

Then $$\lim_{n\to\infty}\frac{f(x)}{g(x)} = \lim_{n\to\infty}\frac{f'(x)}{g'(x)}$$

Eg: $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Let $a_n = \sin\left(\frac{1}{n}\right)$ Let $b_n = \frac{1}{n}$

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\right) \longrightarrow \frac{0}{0}$$

L'Hopital Rule: $\lim_{n\to\infty}\left(\frac{-\frac{1}{n^2}\cos\left(\frac{1}{n}\right)}{-\frac{1}{n^2}}\right)$

$$= \lim_{n\to\infty}\left(\cos\left(\frac{1}{n}\right)\right) = \cos(0) = 1 = L > 0$$

b/c $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (harmonic series), so

$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ diverges!

II Alternating series: (series with some negative terms)

① Alternating series test: AST

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n \quad \text{converges if:}$$

(i) $\lim_{n\to\infty} a_n = 0$

(ii) $a_{n+1} \le a_n$ for all $n \ge 1$

Eg: Alternating harmonic series

Show that $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges!

Solution:

By AST: (i) $a_n = \frac{1}{n}$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$ ✓

(ii) $a_{n+1} = \frac{1}{n+1}$ Clearly, $n+1 \ge n$

$\longrightarrow \frac{1}{n+1} \le \frac{1}{n}$

$\longrightarrow a_{n+1} \le a_n$ ✓

So we can conclude that $\sum (-1)^{n+1} \frac{1}{n}$ converges!

Remark: If $\sum |a_n| \to$, we say that $\sum (-1)^n a_n \to$ absolutely!

(Absolute convergence test!)

Recall that $\sum \frac{1}{n}$ diverges and yet $\sum (-1)^n \frac{1}{n}$ converges! In this case, we say $\sum (-1)^n \frac{1}{n}$ converges conditionally!

Eg: Determine the convergence or divergence of $\sum_{n=1}^{\infty} (-1)^n \frac{n}{2^{n-1}} = -1 + \frac{2}{2^1} - \frac{3}{2^2} + \frac{4}{2^3} - \cdots$

ANS:

(ii): $2^{n-1} \leq 2^n$

$\to \frac{1}{2^n} \leq \frac{1}{2^{n-1}} \to \frac{n+1}{2^n} \leq \frac{n}{2^{n-1}}, \ n \geq 1$

$\to a_{n+1} \leq a_n$ ✓

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n}{2^{n-1}}\right) = \to \frac{\infty}{\infty}$

L'Hopital rule: $\lim_{n\to\infty} \left(\frac{1}{2^{n-1} \ln 2}\right)$ $\qquad (a^u)' = u' a^u \ln a$

$= 0$ ✓

So $\sum_{n=1}^{\infty} (-1)^n \frac{n}{2^{n-1}}$ converges

$\vdots$

Eg: $\sum_{n=0}^{\infty} (-1)^n \frac{n}{n+1}$

Note: $a_n = \frac{n}{n+1}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n}{n+1}\right) = 1 \neq 0 \implies$

$\sum_{n=0}^{\infty} (-1)^n \frac{n}{n+1}$ diverges!

Eg: Does the series $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^2}$ converge absolutely?

Because $\sum \frac{1}{n^2}$ is a p-series with $p = 2 > 1$ converges! $\implies \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^2}$ converges absolutely!